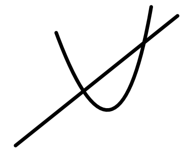
§1.5 Irreducible Components of an algebraic set.

$$V_1 \cup V_2 = \text{alg.}$$

$$V \overset{?}{=} V_1 \cup V_2$$



An algebraic set $V \subset \mathbb{A}^n$ is <u>reducible</u> if $V = V_1 \cup V_2$ (V_i = algebraic sets with $V \neq V_i$)

otherwise V is <u>irreducible</u>.

Example: point

Example: An alg set in $\mathbb{A}^1$ is irr. iff it is a pt or the whole sp.

一般的 alg set 如何判断?

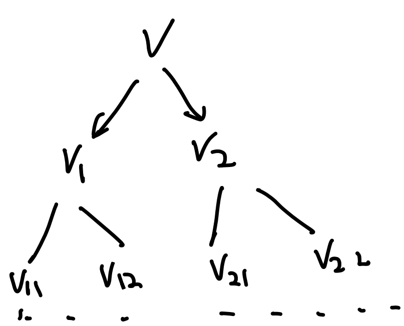
Prop: V = irr $\Leftrightarrow$ $I(V)$ = prime.

Pf: 1° Suppose $I(V) \neq$ prime. $\exists\, F_1, F_2 \notin I(V)$ & $F_1 F_2 \in I(V)$. Then

$$\begin{cases} V = (V \cap V(F_1)) \cup (V \cap V(F_2)) \\ V \neq V \cap V(F_i) \end{cases} \Rightarrow V = \text{reducible}.$$

2° Suppose $V = V_1 \cup V_2$ ($V \neq V_i$). Then $I(V_i) \supsetneqq I(V)$.

$\forall\, F_i \in I(V_i) \setminus I(V)$. $\Rightarrow$ $F_1 F_2 \in I(V) \Rightarrow I(V) \neq$ prime.



有限吗 $\rightsquigarrow$ 代数

Lem: R = noetherian ring.
$\mathcal{S}$ = nonempty collection of ideals in R, Then $\mathcal{S}$ has a maximal member.

Pf: $\mathcal{S}_0 := \mathcal{S}$ $\forall I_0 \in \mathcal{S}_0$ define $\mathcal{S}_i$ & I_i inductively

$$\mathcal{S}_i := \{ I \in \mathcal{S}_{i-1} \mid I \supsetneq I_{i-1} \}$$

$$I_i \in \mathcal{S}_i . \ (\text{if } \mathcal{S}_i \neq \phi).$$

Only need to show $\mathcal{S}_n = \phi$ for some n. Suppose not.

$$I := \bigcup_{n=0}^{\infty} I_n \trianglelefteq R$$

R = noeth. $\Rightarrow$ I = f.g. $I = (F_1, \cdots, F_r)$.

$\Rightarrow \exists\, n \gg 0$ s.t. $F_1, \cdots, F_r \in I_n$

$\Rightarrow I = I_n \Rightarrow I_{n+1} = I_n$ ↯. □

Cor: any collection of algebraic sets in $\mathbb{A}^n(k)$ has a minimal member

Pf $\{V_\alpha\} \rightarrow \{ I(V_\alpha) \trianglelefteq k[x_1, \cdots, x_n] \}$ □

Thm: $V \subseteq \mathbb{A}^n(k)$ algebraic set. Then $\exists !$ irr. algebraic sets $V_1, \cdots, V_m$ s.t.

$$V = V_1 \cup \cdots \cup V_m$$

and $V_i \not\subseteq V_j$ for all $i \neq j$. V_i's are called the irreducible components of V.

pf: Existance.

$\mathcal{S} := \{ V \subset \mathbb{A}^n(k) \mid$ alg. sets not union of f. irr. ones $\}$

WNTS: $\mathcal{S} = \phi$. Suppose Not. Let V be a minimal one.

$V \neq$ irr $\Rightarrow V = V_1 \cup V_2 \quad (V_1 \neq V \neq V_2)$

$\Rightarrow V_1, V_2 \notin \mathcal{S} \Rightarrow \begin{cases} V_1 = \bigcup V_{i_1} \\ V_2 = \bigcup V_{i_2} \end{cases}$

$\Rightarrow V = \left(\bigcup V_{i_1}\right) \cup \left(\bigcup V_{i_2}\right) \quad ↯$

throw away V_i s.t. $V_i \subset V_j$ $(i \neq j)$

uniqueness. $V_1 \cup \cdots \cup V_m = W_1 \cup \cdots \cup W_n$

$\Rightarrow \forall i,\ V_i = (W_1 \cap V_i) \cup \cdots \cup (W_n \cap V_i)$

$\Rightarrow \exists j$ s.t. $V_i = W_j \cap V_i$ (i.e. $V_i \subseteq W_j$)

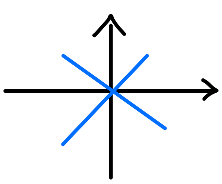
Conversely, $\exists i'$ s.t. $W_j \subseteq V_{i'}$

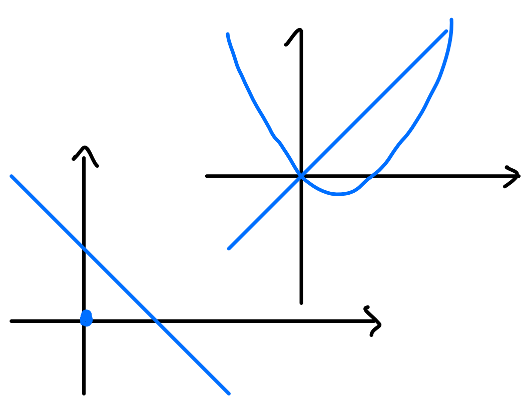
$\Rightarrow i = i'$ & $W_j = V_i$.

$\Rightarrow \ldots$

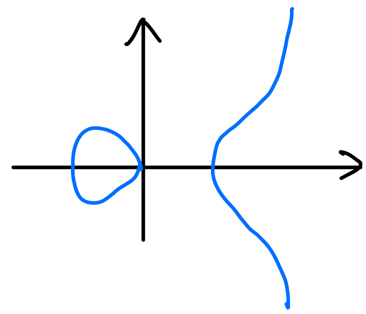
$\square$

Example:



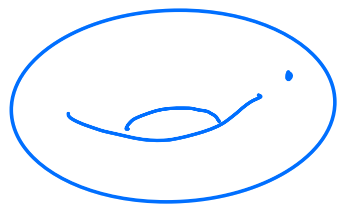


Example: $V(y^2 - x(x^2-1)) \subseteq \mathbb{A}^2(\mathbb{R})$



$\mathbb{R}$

$V(y^2 - x(x^2-1)) \subseteq \mathbb{A}^2(\mathbb{C})$



$\mathbb{C}$

Rmk: $I = \text{prime} \not\Rightarrow V(I) = \text{irr}$! 域太小,导致点太少.